Non-contractability and revenge*

Julien Murzi[†] & Lorenzo Rossi[‡]

August 16, 2018

Abstract

It is often argued that fully structural theories of truth and related notions are incapable of expressing a nonstratified notion of defectiveness. We argue that recently much-discussed *non-contractive theories* suffer from the same expressive limitation, provided they identify the defective sentences with the sentences that yield triviality if they are assumed to satisfy structural contraction.

Keywords: Semantic paradoxes · Contraction · Revenge

Here's a standard recipe for revenge. Faced with paradoxes such as the Liar and Curry, the non-classical theorist constructs a theory of truth S that non-trivially expresses truth, in spite of Tarski's Theorem. More precisely, the theorist shows that S can be non-trivially closed under (at least) the naïve principles Tr-R and Tr-L:

$$\frac{\Gamma \vdash \varphi, \Delta}{\Gamma \vdash \mathsf{Tr}(\ulcorner \varphi \urcorner), \Delta}\ \text{Tr-R} \qquad \frac{\Gamma, \varphi \vdash \Delta}{\Gamma, \mathsf{Tr}(\ulcorner \varphi \urcorner) \vdash \Delta}\ \text{Tr-L}\ ,$$

where $\ulcorner \varphi \urcorner$ is a name of φ, and Γ and Δ range over multisets of sentences.[1] The reason why S can be non-trivial is simple enough: intuitively paradoxical sentences such as the Liar sentence (a sentence asserting its untruth) don't satisfy all the principles of classical logic in S, whence the paradoxical reasonings they give rise to break down. In her next step, the revenger identifies a property Φ of sentences, intuitively expressing some notion of *paradoxicality*, where a sentence φ is paradoxical just in case absurdity follows in S from the assumption that φ satisfies all the principles of classical logic. The revenger now defines a sentence ρ attributing to itself the property of being Φ. She then establishes via Liar-like reasoning that ρ trivialises S if it satisfies all the principles of classical logic and that, for this reason, ρ must be paradoxical, thus establishing ρ. But, the revenger reasons, if S was correctly set up, S only derives sentences that are not paradoxical, whence ρ must be not Φ. Contradiction.[2]

*Many thanks to Barteld Kooi, Francesco Paoli, Lionel Shapiro, and Elia Zardini for helpful discussion along the way. Special thanks to Dave Ripley and two anonymous referees for comments on previous drafts that led to significant improvements. This research was supported by the Alexander von Humboldt Foundation and the FWF (grant number P29716), whose support we gratefully acknowledge.

[†]Philosophy Department (KGW), University of Salzburg, and Munich Centre for Mathematical Philosophy, Ludwig-Maximilians University, julien.murzi@sbg.ac.at

[‡]Philosophy Department (KGW), University of Salzburg, lorenzo.rossi@sbg.ac.at

[1]Multisets are just like sets, except that repetitions count. For instance, $\{a, a\}$ and $\{a\}$ are the same set, but $[a, a]$ and $[a]$ are different multisets (we represent multisets by means of square brackets).

[2]See Priest (2007, p. 226). For recent discussion on revenge, see e.g. Beall (2007b); Shapiro (2010); Scharp (2013). We should note that the revenge recipe just sketched only applies to consistent theories. However, it can be modified so as to also cover inconsistent approaches (see [REDACTED]).

In this paper, we argue that a version of the strategy applies to a wide family of *non-contractive* theories, i.e. theories which reject the left and right structural rules of *contraction*:

$$\dfrac{\Gamma, \varphi, \varphi \vdash \Delta}{\Gamma, \varphi \vdash \Delta}\,\text{LContr} \qquad\qquad \dfrac{\Gamma \vdash \varphi, \varphi, \Delta}{\Gamma \vdash \varphi, \Delta}\,\text{RContr},$$

while keeping the other standard structural rules, namely *reflexivity*, *weakening* (left and right), and *cut*:

$$\dfrac{}{\varphi \vdash \varphi}\,\text{SRef} \qquad \dfrac{\Gamma \vdash \Delta}{\Gamma, \varphi \vdash \Delta}\,\text{LWeak} \qquad \dfrac{\Gamma \vdash \Delta}{\Gamma \vdash \varphi, \Delta}\,\text{RWeak}$$

$$\dfrac{\Gamma \vdash \varphi, \Delta \qquad \Gamma', \varphi \vdash \Delta'}{\Gamma, \Gamma' \vdash \Delta, \Delta'}\,\text{Cut}$$

Non-contractive theories have long been advocated in the context of revisionary treatments of the semantic paradoxes, largely in virtue of their proof-theoretic elegance (see e.g. Fitch, 1942, 1948). More recently, they have been claimed to be superior to standard paracomplete and paraconsistent non-classical approaches, on the grounds that, unlike them, they can handle paradoxes of naïve logical properties (Shapiro (2011); Zardini (2015); [REDACTED]).[3] Whatever their relative merits over standard revisionary approaches, we argue that they suffer from essentially the same expressive limitations.

Here's our plan. §1 rehearses the non-contractive approach to paradox. §2 introduces the notions of contractability and contractable truth. §§3-4 present a revenge argument for non-contractive theories. §5 concludes.

1 Naïve truth, contraction-freedom, and classical recapture

Let S be a theory that interprets a *modicum* of arithmetic, is formulated in classical logic, and is closed under the naïve truth rules Tr-R and Tr-L. Let λ be a sentence—a Liar sentence—provably equivalent to $\neg\mathsf{Tr}(\ulcorner\lambda\urcorner)$, and let negation be governed by its standard classical rules:

$$\dfrac{\Gamma, \varphi \vdash \Delta}{\Gamma \vdash \neg\varphi, \Delta}\,\neg\text{-R} \qquad\qquad \dfrac{\Gamma \vdash \varphi, \Delta}{\Gamma, \neg\varphi \vdash \Delta}\,\neg\text{-L}$$

It can now be easily established that S is trivial.[4] One first proves that S derives the sequent $\mathsf{Tr}(\ulcorner\lambda\urcorner) \vdash \emptyset$—call this derivation $\mathcal{D}_0$:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\lambda \vdash \lambda}\,\text{SRef}}{\mathsf{Tr}(\ulcorner\lambda\urcorner) \vdash \lambda}\,\text{Tr-L}}{\mathsf{Tr}(\ulcorner\lambda\urcorner) \vdash \neg\mathsf{Tr}(\ulcorner\lambda\urcorner)}\,\text{Def. of }\lambda \qquad \dfrac{\dfrac{}{\mathsf{Tr}(\ulcorner\lambda\urcorner) \vdash \mathsf{Tr}(\ulcorner\lambda\urcorner)}\,\text{SRef}}{\mathsf{Tr}(\ulcorner\lambda\urcorner), \neg\mathsf{Tr}(\ulcorner\lambda\urcorner) \vdash}\,\neg\text{-L}}{\mathsf{Tr}(\ulcorner\lambda\urcorner), \mathsf{Tr}(\ulcorner\lambda\urcorner) \vdash}\,\text{Cut}}{\mathsf{Tr}(\ulcorner\lambda\urcorner) \vdash}\,\text{LContr}$$

[3]For general background on paracomplete approaches, see e.g. Kripke (1975), Field (2007, 2008), Horsten (2009); on paraconsistent approaches, see e.g. Asenjo and Tamburino (1975), Goodship (1996), Priest (2006), Beall (2009, 2011). We should also note that, among non-contractive theorists, Elia Zardini has explicitly acknowledged that the handling of the paradoxes of naïve logical properties comes with what may be regarded as a cost, viz. that the meta-theory must itself be non-classical, and indeed substructural (Zardini, 2013, 2014). We return to this point in §2 below.

[4]We implicitly make use of a rule of intersubstitutivity of equivalents for sentences, here and throughout. Nothing crucial hinges on this choice.

One now uses two copies of $\mathcal{D}_0$ to derive the empty sequent:

$$\dfrac{\dfrac{\dfrac{\dfrac{\begin{matrix}\mathcal{D}_0\\ \mathsf{Tr}(\ulcorner\lambda\urcorner)\vdash\end{matrix}}{\vdash\neg\mathsf{Tr}(\ulcorner\lambda\urcorner)}\neg\text{-R}}{\vdash\lambda}\text{Def. of }\lambda}{\vdash\mathsf{Tr}(\ulcorner\lambda\urcorner)}\text{Tr-R}\qquad\begin{matrix}\mathcal{D}_0\\ \mathsf{Tr}(\ulcorner\lambda\urcorner)\vdash\end{matrix}}{\vdash}\text{Cut}$$

In presence of the weakening rules, every sentence is now entailed by any sentence. This is the Liar Paradox.[5]

A number of authors have recently, and not so recently, suggested blaming structural contraction as the culprit of the Liar, and of semantic paradoxes in general (Fitch, 1942, 1948; Shapiro, 2011; Zardini, 2011; Mares and Paoli, 2014). In particular, Elia Zardini (2011) proves consistency for a non-contractive naïve theory of truth and naïve logical properties, validating naïve truth-principles such as Tr-R and Tr-L.[6] The propositional fragment of the logic of the theory is multiplicative affine linear logic (henceforth, WMLL)—a logic validating SRef, LWeak, RWeak, and Cut, but not LContr and RContr.

Negation and the conditional are interpreted the standard way. For completeness, here are the rules for $\rightarrow$:

$$\dfrac{\Gamma,\varphi\vdash\psi,\Delta}{\Gamma\vdash\varphi\rightarrow\psi,\Delta}\rightarrow\text{-R}\qquad\qquad\dfrac{\Gamma\vdash\varphi,\Delta\qquad\Gamma',\psi\vdash\Delta'}{\Gamma,\Gamma',\varphi\rightarrow\psi\vdash\Delta,\Delta'}\rightarrow\text{-L}$$

Conjunction and disjunction are interpreted, respectively, by the multiplicative connectives $\otimes$ and $\oplus$. Here are the rules for $\otimes$:

$$\dfrac{\Gamma\vdash\varphi,\Delta\qquad\Gamma'\vdash\psi,\Delta'}{\Gamma,\Gamma'\vdash\varphi\otimes\psi,\Delta,\Delta'}\otimes\text{-R}\qquad\qquad\dfrac{\Gamma,\varphi,\psi\vdash\Delta}{\Gamma,\varphi\otimes\psi\vdash\Delta}\otimes\text{-L}$$

And here are the rules for $\oplus$:

$$\dfrac{\Gamma\vdash\varphi,\psi,\Delta}{\Gamma\vdash\varphi\oplus\psi,\Delta}\oplus\text{-R}\qquad\qquad\dfrac{\Gamma,\varphi\vdash\Delta\qquad\Gamma',\psi\vdash\Delta'}{\Gamma,\Gamma',\varphi\oplus\psi\vdash\Delta,\Delta'}\oplus\text{-L}$$

Absent SContr, the rules yield a distinctively non-classical interpretations of 'and' and 'or'. For one thing, in keeping with the rejection of LContr and RContr, φ and $\varphi\otimes\varphi$ have different logical strength: conjunction is not idempotent and φ and $\varphi\otimes\varphi$ are not in general equivalent. For another, while $\oplus$ satisfies the Law of Excluded Middle

$$\dfrac{}{\vdash\varphi\oplus\neg\varphi}\text{LEM},$$

it only satisfies *weak proof by cases* (Zardini, 2011, p. 516):

$$\dfrac{\Gamma\vdash\varphi\oplus\psi,\Delta\qquad\Gamma',\varphi\vdash\gamma,\Delta'\qquad\Gamma'',\psi\vdash\delta,\Delta''}{\Gamma,\Gamma',\Gamma''\vdash\gamma\oplus\delta,\Delta,\Delta',\Delta''}\text{WPC}$$

[5]We note that the present version of the Liar does not rely on the Law of Excluded Middle or similar classical principles.

[6]Zardini's theory actually validates a much stronger form of naïveté, viz. the intersubstitutivity *salva veritate* of φ and $\mathsf{Tr}(\ulcorner\varphi\urcorner)$ in all non-opaque contexts.

A multiplicative disjunction only entails the disjunction of whatever its disjuncts separately entail.

However, in spite of the non-classicality of $\otimes$ and $\oplus$, WMLL need not be thought as radical. In the multiplicative setting Zardini favours, full classical reasoning about φ can be *recaptured* whenever φ satisfies both $\varphi \to (\varphi \otimes \varphi)$ and $(\varphi \oplus \varphi) \to \varphi$.

Theorem 1 (Zardini (2011), Theorem 3.19)**.** *Let S be any theory with language $\mathcal{L}_S$ with underlying logic at least as strong as* WMLL*. Then, for any $\varphi \in \mathcal{L}_S$, φ satisfies* LContr *and* RContr *if and only if it satisfies both $\varphi \to (\varphi \otimes \varphi)$ and $(\varphi \oplus \varphi) \to \varphi$.*

In particular, $\varphi \to (\varphi \otimes \varphi)$ and $(\varphi \oplus \varphi) \to \varphi$ are, respectively, LContr- and RContr-*recapturing*, in the sense specified by the following fact:

Fact 2. *Let S be any theory with language $\mathcal{L}_S$ with underlying logic at least as strong as* WMLL*. Then, for any $\varphi \in \mathcal{L}_S$, φ satisfies* LContr *if it satisfies $\varphi \to (\varphi \otimes \varphi)$ and φ satisfies* RContr *if it satisfies $(\varphi \oplus \varphi) \to \varphi$.*

Proof. The proof makes use of the following weaker versions of LContr and RContr, both of which are derivable in WMLL:

$$\frac{\Gamma, \varphi, \varphi \vdash \Delta}{\Gamma, \varphi, \varphi \to (\varphi \otimes \varphi) \vdash \Delta}\ \text{LContr}_\text{W} \qquad \frac{\Gamma \vdash \varphi, \varphi, \Delta}{\Gamma, (\varphi \oplus \varphi) \to \varphi \vdash \varphi, \Delta}\ \text{RContr}_\text{W}$$

The derivability of LContr_W and RContr_W is respectively established by the following derivations:

$$\frac{\dfrac{}{\varphi \vdash \varphi}\ \text{SRef} \qquad \dfrac{\Gamma, \varphi, \varphi \vdash \Delta}{\Gamma, \varphi \otimes \varphi \vdash \Delta}\ \otimes\text{-L}}{\Gamma, \varphi, \varphi \to (\varphi \otimes \varphi) \vdash \Delta}\ \to\text{-L} \qquad \frac{\dfrac{\Gamma \vdash \varphi, \varphi, \Delta}{\Gamma \vdash \varphi \oplus \varphi, \Delta}\ \oplus\text{-R} \qquad \dfrac{}{\varphi \vdash \varphi}\ \text{SRef}}{\Gamma, (\varphi \oplus \varphi) \to \varphi \vdash \varphi, \Delta}\ \to\text{-L}$$

We now prove that LContr holds given $\varphi \to (\varphi \otimes \varphi)$ and LContr_W:

$$\frac{\vdash \varphi \to (\varphi \otimes \varphi) \qquad \dfrac{\Gamma, \varphi, \varphi \vdash \Delta}{\Gamma, \varphi, \varphi \to (\varphi \otimes \varphi) \vdash \Delta}\ \text{LContr}_\text{W}}{\Gamma, \varphi \vdash \Delta}\ \text{Cut}$$

An analogous derivation establishes that RContr holds given $(\varphi \oplus \varphi) \to \varphi$ and RContr_W. □

2 Contractability and contractable truth

Let WMLLTT be the result of closing a sufficiently expressive theory whose underlying logic is WMLL under Tr-R and Tr-L. Then, it is a fact about WMLLTT that sentences such as λ satisfy LContr or RContr only on pain of triviality. That is, these sentences are *non-contractable*.[7]

[7] How can a restriction of contraction be plausibly motivated? As John Myhill once put it, 'while [restricting contraction] is *proof-theoretically* natural [...] no form known to us is *philosophically* natural' (Myhill, 1975, p. 182). In a number of papers, Zardini has recently sought to justify restrictions of LContr and RContr on the grounds that sentences such as λ are unstable where, according to Zardini, a sentence φ is unstable if and only if there is a ψ such that the state-of-affairs expressed by φ leads to the state-of-affairs expressed by ψ and those two states-of-affairs are incompatible—see Zardini (2015, p. 492) and, especially, Zardini (2011, pp. 503-506). But how to more precisely interpret Zardini's notion of instability? As Zardini puts it, the notion of an unstable state-of-affairs involves stepping 'out of the abstract realm of *formal* theories of truth ... [to] engage in some concrete *metaphysics* of truth' (Zardini, 2011, p. 504). Here we briefly note that, from a purely logical point of view, Zardini's notion of instability is difficult to make precise. It might be

Non-contractability so understood gives rise to a version of the Knower Paradox (Kaplan and Montague, 1960; Myhill, 1960), involving a sentence κ provably equivalent to $\neg\mathsf{Ct}(\ulcorner\kappa\urcorner)$, where $\mathsf{Ct}(x)$ is a predicate expressing contractable truth. That is, κ says of itself that it is not true and contractable, just like the Liar sentence says of itself that it is not true. The paradox is effectively a variant of the Liar Paradox, and it is unsurprisingly invalid in non-contractive theories. However, we argue in §4 that non-contractive theorists are committed to the claim that κ is non-contractable, which in turn triggers a version of the revenge recipe we started with.

First off, some background on contractability and related notions. Theorem 1 motivates the following rules for a contractability operator: that if φ satisfies $\varphi \to (\varphi \otimes \varphi)$ and $(\varphi \oplus \varphi) \to \varphi$, then φ is contractable; and that if Δ is derivable from the assumption (represented by $\varphi \to (\varphi \otimes \varphi)$ or $(\varphi \oplus \varphi) \to \varphi$) that one can left or right contract on φ, then Δ also follows from the assumption that φ is contractable. In symbols, where C is an operator expressing contractability:

$$\frac{\Gamma \vdash \varphi \to (\varphi \otimes \varphi), \Delta \qquad \Gamma' \vdash (\varphi \oplus \varphi) \to \varphi, \Delta'}{\Gamma, \Gamma' \vdash \mathsf{C}(\varphi), \Delta, \Delta'}\ \text{C-R}$$

$$\frac{\Gamma, \varphi \to (\varphi \otimes \varphi) \vdash \Delta}{\Gamma, \mathsf{C}(\varphi) \vdash \Delta}\ \text{C-L}_1 \qquad \frac{\Gamma, (\varphi \oplus \varphi) \to \varphi \vdash \Delta}{\Gamma, \mathsf{C}(\varphi) \vdash \Delta}\ \text{C-L}_2$$

The rules can be generalised as follows. Let $[\varphi]^n$ be the multiset consisting of n occurrences of φ. Moreover, let us assume that Γ in C-L_1^+ and C-L_2^+ below does not contain instances of, respectively, $\varphi \to (\varphi \otimes \varphi)$ and $(\varphi \oplus \varphi) \to \varphi$, and let $m \geq 1$. Then, one can formulate the following general rules for introducing $\mathsf{C}(\varphi)$ on the left:[8]

$$\frac{\Gamma, [\varphi \to (\varphi \otimes \varphi)]^m \vdash \Delta}{\Gamma, \mathsf{C}(\varphi) \vdash \Delta}\ \text{C-L}_1^+ \qquad \frac{\Gamma, [(\varphi \oplus \varphi) \to \varphi]^m \vdash \Delta}{\Gamma, \mathsf{C}(\varphi) \vdash \Delta}\ \text{C-L}_2^+$$

C-L_1^+ says that if Δ is derivable from the assumption that φ satisfies m contractions (represented by $[(\varphi \to (\varphi \otimes \varphi)]^m)$, then Δ is derivable from the assumption that φ is left contractable. Similarly for C-L_2^+. The rationale behind the rules is that, by non-contractive lights, structural contraction, whether left or right, is the source of the paradoxes. To see this, consider the case where Δ is the empty set in all the above left rules. Then, C-L_1^+ and C-L_2^+ say that φ cannot be contracted on if contracting on φ, *it doesn't matter how many times*, yields the empty set, and therefore (by weakening) *any* sentence.

To be sure, the move from {C-L_1, C-L_2} to {C-L_1^+, C-L_2^+} is not altogether innocent. As a referee observed, C-L_1^+ and C-L_2^+ are derivable from C-L_1 and C-L_2 only if SContr is available. Otherwise, the best one can do (applying C-L_1 and C-L_2 m times) is

understood meta-theoretically as the claim that, for some suitable non-contractive theory S with language $\mathcal{L}_S$, $\varphi \vdash_S \psi$ but $\nvdash_S \varphi \otimes \psi$. But such a reading would appear to be too strong. For suppose S interprets a *modicum* of arithmetic and let γ be a Gödel sentence for S. Now consider one of γ's consequences, such as $\gamma \oplus \gamma$. Then, since $\nvdash_S \gamma$, it follows that $\nvdash_S \gamma \otimes (\gamma \oplus \gamma)$ also holds. And since $\gamma \vdash_S \gamma \oplus \gamma$, the sentence γ winds up being unstable. Yet this appears to be problematic: if we let S be Peano Arithmetic, γ is an arithmetical truth! Alternatively, it might be thought that φ is unstable if and only if $\varphi \vdash_S \psi$ and $\varphi, \psi \vdash_S$. But such a reading is also too strong. Since $\neg t = t \vdash_S t = t$ and $\neg t = t, t = t \vdash_S$, it follows that $\neg t = t$ also winds up being unstable. Yet, intuitively, $\neg t = t$ is simply false, since it is false in every model validating the classical theory of identity, and would rather appear to be stable. For these reasons, in constructing a revenge argument for non-contractive approaches to paradox, we focus on the minimal notion of non-contractability discussed in the main text. Zardini's notion of instability is further discussed in Zardini (2018).

[8] A related principle is used in [REDACTED] to run a different argument against non-contractive approaches.

$$\dfrac{\Gamma, [\varphi \to (\varphi \otimes \varphi)]^m \vdash \Delta}{\Gamma, \underbrace{\mathsf{C}(\varphi), \ldots, \mathsf{C}(\varphi)}_{m\text{-}times} \vdash \Delta}\ \text{C-L}_1^m \qquad \dfrac{\Gamma, [(\varphi \oplus \varphi) \to \varphi]^m \vdash \Delta}{\Gamma, \underbrace{\mathsf{C}(\varphi), \ldots, \mathsf{C}(\varphi)}_{m\text{-}times} \vdash \Delta}\ \text{C-L}_2^m$$

But, it might be objected, the non-contractive theorist who rejects contraction *in all its forms* has a reason to reject contracting on sentences of the form $\mathsf{C}(\varphi)$, and hence of resting the move from {C-L$_1$, C-L$_2$} to {C-L$_1^+$, C-L$_2^+$}.

However, C-L$_1^m$ and C-L$_2^m$ are unacceptable by non-contractive lights, since they would commit the non-contractive theorist to an untenable conception of paradoxicality. More precisely, they would commit such a theorist to distinguishing between different numbers of applications of SContr in a derivation, which would sit poorly with her diagnosis of what goes wrong in paradoxical derivations. According to non-contractive wisdom, *indiscriminate uses* of SContr must be rejected *in general*. That is, non-contractive theorists disallow the following generalised version of SContr:

$$\dfrac{\Gamma, [\varphi]^j \vdash \Delta}{\Gamma, [\varphi]^i \vdash \Delta}\ \text{SContr}^\star \quad (\text{where } j > i),$$

according to which, if Δ follows from Γ and i occurrences of φ, then Δ follows from Γ and at least one occurrence of φ. The idea that if SContr* applied to φ leads to $\bot$ then φ is non-contractable is at the heart of the non-contractive approach to semantic paradox: one must disallow *whatever number* of applications of SContr to φ lead to $\bot$ in a paradoxical derivation. This is captured by the rules C-L$_1^+$ and C-L$_2^+$, but cannot be expressed by the non-contractive theorist who expresses non-contractability by means of rules of the form C-L$_1^m$ and C-L$_2^m$. In keeping with Fact 2, let m-contractions on φ be represented by m-many instances of $\varphi \to (\varphi \otimes \varphi)$ or $(\varphi \oplus \varphi) \to \varphi)$. Then, the non contractive theorist who accepts C-L$_1^m$ and C-L$_2^m$ but rejects C-L$_1^+$ and C-L$_2^+$ can only express that if m-many contractions on φ lead to absurdity, then m-many claims of the form $\mathsf{C}(\varphi)$ lead to absurdity. In effect, this would be tantamount to introducing a denumerable infinity of contractability operators, each of which expresses k-contractability, for every positive integer k. However, it is clear that, on such a view, the non-contractive theorist would be prevented from blaming, as she does, *contraction in general* as a source of the paradoxes. Indeed, she would not be in a position to express contractability in general—in keeping with the results to be presented in §4.[9]

Now say that φ is *contractably true* if and only if both φ and $\mathsf{C}(\varphi)$ hold. More formally:

(CT) $\vdash \mathsf{Ct}(\ulcorner \varphi \urcorner) \leftrightarrow (\varphi \otimes \mathsf{C}(\varphi))$

It immediately follows that, in any theory validating CT, contractable truth is factive:

(FACT) $\mathsf{Ct}(\ulcorner \varphi \urcorner) \vdash \varphi$

It can be further established that a theory S is closed under C-R only if it is also closed under the following necessitation-like rule:

(NEC$_\mathsf{C}$) If $\Gamma \vdash \varphi, \Delta$, then $\Gamma, \Gamma \vdash \mathsf{C}(\varphi), \Delta, \Delta$.

If Γ and Δ are empty, NEC$_\mathsf{C}$ yields the standard rule of necessitation, that if $\vdash \varphi$, then $\vdash \mathsf{C}(\varphi)$.

[9] We are grateful to [REDACTED] for valuable discussion on this point.

Fact 3. *Let S be any non-contractive theory with consequence relation $\vdash$ with underlying logic at least as strong as* WMLL. *Then, S is closed under* C-R *only if it is closed under* $\mathsf{NEC_C}$.

Proof. We reason in S, assuming that φ, Δ is derivable from Γ. One first derives

$$\Gamma \vdash \varphi \rightarrow (\varphi \otimes \varphi), \Delta$$

from $\Gamma \vdash \varphi, \Delta$:

$$\dfrac{\dfrac{\Gamma \vdash \varphi, \Delta \qquad \dfrac{}{\varphi \vdash \varphi}\text{SRef}}{\Gamma, \varphi \vdash \varphi \otimes \varphi, \Delta}\otimes\text{-R}}{\Gamma \vdash \varphi \rightarrow (\varphi \otimes \varphi), \Delta}\rightarrow\text{-R}$$

One then notices that $\Gamma \vdash (\varphi \oplus \varphi) \rightarrow \varphi, \Delta$ is also derivable from $\Gamma \vdash \varphi, \Delta$:

$$\dfrac{\dfrac{\Gamma \vdash \varphi, \Delta}{\Gamma, \varphi \oplus \varphi \vdash \varphi, \Delta}\text{LWeak}}{\Gamma \vdash (\varphi \oplus \varphi) \rightarrow \varphi, \Delta}\rightarrow\text{-R}$$

Putting the two pieces together, the sequent $\Gamma, \Gamma \vdash \mathsf{C}(\varphi), \Delta, \Delta$ follows by C-R.[10] □

3 The Non-contractability Knower

One can now generate a version of the Knower Paradox, call it the Non-contractability Knower, involving a sentence κ provably equivalent to $\neg\mathsf{Ct}(\ulcorner\kappa\urcorner)$. Informally, we may reason thus. One assumes $\mathsf{Ct}(\ulcorner\kappa\urcorner)$, derives κ via FACT, whence $\neg\mathsf{Ct}(\ulcorner\kappa\urcorner)$ by definition of κ. Assuming again $\mathsf{Ct}(\ulcorner\kappa\urcorner)$, one must now discharge both instances of $\mathsf{Ct}(\ulcorner\kappa\urcorner)$ and conclude $\neg\mathsf{Ct}(\ulcorner\kappa\urcorner)$ by $\neg$-R . Next, one derives κ by construction of κ, whence $\mathsf{C}(\kappa)$ courtesy of $\mathsf{NEC_C}$. Repeating again the derivation of κ, κ and $\mathsf{C}(\kappa)$ now yield $\mathsf{Ct}(\ulcorner\kappa\urcorner)$. Contradiction.

Much like in the case of the Liar, the paradox yields a result to the effect that contractable truth is undefinable in S if LContr holds.

Definition 4. A theory S *defines contractable truth* if it is closed under $\mathsf{NEC_C}$, C-L_1^+, C-L_2^+, and CT.

While we don't think that each of $\mathsf{NEC_C}$, C-L_1^+, C-L_2^+, and CT is unassailable, they arguably jointly characterise an intuitive, if naïve, notion of non-contractability. $\mathsf{NEC_C}$ is provable from from C-R, which is in turn justified by Theorem 1 (as shown in Fact 3). As for C-L_1^+ and C-L_2^+, we have seen in §2 that they directly fall out of the non-contractive diagnosis of the paradoxes. More precisely, when Γ and Δ are empty, they tell us that if an arbitrary number of contractions on φ yields triviality, then φ is not contractable. Finally, CT simply employs the operator characterised by $\mathsf{NEC_C}$, C-L_1^+, and C-L_2^+ to form a *predicate* expressing contractable truth. We now show that contractable truth is undefinable.

[10] Jc Beall (2006) suggests that the intuitive defectiveness of paradoxical sentences may be expressed by an operator PN expressing *paranormality*. He then notices that one could define a factive notion of robust truth $\mathbb{T}(x)$ by setting $\mathbb{T}(\ulcorner\varphi\urcorner) \leftrightarrow (\varphi \wedge \neg\mathsf{PN}(\varphi))$. However, he also insists that $\mathbb{T}(x)$ *may not satisfy necessitation*. That is, one should not expect that, if $\vdash \varphi$, then $\vdash \mathbb{T}(\ulcorner\varphi\urcorner)$. As a result, some paranormal sentences are true, 'but not thereby robustly true' (Beall, 2007a, §4.2). It is a consequence of Fact 3 that Beall's strategy doesn't apply to theories closed under C-R.

Proposition 5. *Let S be any theory with language $\mathcal{L}_S$ strong enough to prove the existence of a sentence κ equivalent to $\neg\mathsf{Ct}(\ulcorner\kappa\urcorner)$, with underlying logic at least as strong as* WMLL. *Let $\vdash$ be S's consequence relation and suppose S defines contractable truth. Then, S is closed under* LContr *only if it is trivial.*

Proof. Let κ be a sentence of $\mathcal{L}_S$ provably equivalent to $\neg\mathsf{Ct}(\ulcorner\kappa\urcorner)$. One first derives $\mathsf{Ct}(\ulcorner\kappa\urcorner)\vdash$ contracting on $\mathsf{Ct}(\ulcorner\kappa\urcorner)$—call this derivation $\mathcal{D}_1$:

$$\dfrac{\dfrac{\dfrac{\dfrac{}{\mathsf{Ct}(\ulcorner\kappa\urcorner)\vdash\kappa}\ \mathsf{FACT}}{\mathsf{Ct}(\ulcorner\kappa\urcorner)\vdash\neg\mathsf{Ct}\ulcorner\kappa\urcorner}\ \text{Def. of }\kappa \qquad \dfrac{\dfrac{}{\mathsf{Ct}(\ulcorner\kappa\urcorner)\vdash\mathsf{Ct}(\ulcorner\kappa\urcorner)}\ \mathsf{SRef}}{\neg\mathsf{Ct}(\ulcorner\kappa\urcorner),\mathsf{Ct}(\ulcorner\kappa\urcorner)\vdash}\ \neg\text{-L}}{\mathsf{Ct}(\ulcorner\kappa\urcorner),\mathsf{Ct}(\ulcorner\kappa\urcorner)\vdash}\ \mathsf{Cut}}{\mathsf{Ct}(\ulcorner\kappa\urcorner)\vdash}\ \mathsf{LContr}$$

Three copies of $\mathcal{D}_1$ can now be turned into a proof of the empty sequent:[11]

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\overset{\mathcal{D}_1}{\mathsf{Ct}(\ulcorner\kappa\urcorner)\vdash}}{\vdash\neg\mathsf{Ct}(\ulcorner\kappa\urcorner)}\ \neg\text{-R}}{\vdash\kappa}\ \text{Def. of }\kappa}{\vdash\mathsf{C}(\kappa)}\ \mathsf{NEC_C} \qquad \dfrac{\dfrac{\overset{\mathcal{D}_1}{\mathsf{Ct}(\ulcorner\kappa\urcorner)\vdash}}{\vdash\neg\mathsf{Ct}(\ulcorner\kappa\urcorner)}\ \neg\text{-R}}{\vdash\kappa}\ \text{Def. of }\kappa}{\vdash\mathsf{Ct}(\ulcorner\kappa\urcorner)}\ \mathsf{CT} \qquad \overset{\mathcal{D}_1}{\mathsf{Ct}(\ulcorner\kappa\urcorner)\vdash}}{\vdash}\ \mathsf{Cut}$$

□

This is the Non-contractability Knower.

To be sure, a natural non-contractivist response is to oserve that $\mathsf{Ct}(\ulcorner\kappa\urcorner)$ is non-contractable, and disallow, for this reason, left contracting on $\mathsf{Ct}(\ulcorner\kappa\urcorner)$ in $\mathcal{D}_1$. The response is indeed available to the the non-contractive theorist, who can *prove* that $\mathsf{Ct}(\ulcorner\kappa\urcorner)$ is non-contractable.

Proposition 6. *Let S be any theory with language $\mathcal{L}_S$ strong enough to prove the existence of a sentence κ equivalent to $\neg\mathsf{Ct}(\ulcorner\kappa\urcorner)$ with underlying logic at least as strong as* WMLL. *Let $\vdash$ be S's consequence relation and suppose S defines contractable truth. Then, S proves $\vdash\neg\mathsf{C}(\mathsf{Ct}(\ulcorner\kappa\urcorner))$.*

Proof. Let κ be a sentence of $\mathcal{L}_S$ provably equivalent to $\neg\mathsf{Ct}(\ulcorner\kappa\urcorner)$, and let $\mathsf{LC}_{\mathsf{Ct}(\ulcorner\kappa\urcorner)}$ be shorthand for

$$\mathsf{Ct}(\ulcorner\kappa\urcorner)\rightarrow(\mathsf{Ct}(\ulcorner\kappa\urcorner)\otimes\mathsf{Ct}(\ulcorner\kappa\urcorner)).$$

One first derives $\mathsf{LC}_{\mathsf{Ct}(\ulcorner\kappa\urcorner)},\mathsf{Ct}(\ulcorner\kappa\urcorner)\vdash\varnothing$—call this derivation $\mathcal{D}_2$:

$$\dfrac{\dfrac{\dfrac{\dfrac{}{\mathsf{Ct}(\ulcorner\kappa\urcorner)\vdash\kappa}\ \mathsf{FACT}}{\mathsf{Ct}(\ulcorner\kappa\urcorner)\vdash\neg\mathsf{Ct}\ulcorner\kappa\urcorner}\ \text{Def. of }\kappa \qquad \dfrac{\dfrac{}{\mathsf{Ct}(\ulcorner\kappa\urcorner)\vdash\mathsf{Ct}(\ulcorner\kappa\urcorner)}\ \mathsf{SRef}}{\neg\mathsf{Ct}(\ulcorner\kappa\urcorner),\mathsf{Ct}(\ulcorner\kappa\urcorner)\vdash}\ \neg\text{-L}}{\mathsf{Ct}(\ulcorner\kappa\urcorner),\mathsf{Ct}(\ulcorner\kappa\urcorner)\vdash}\ \mathsf{Cut}}{\mathsf{LC}_{\mathsf{Ct}(\ulcorner\kappa\urcorner)},\mathsf{Ct}(\ulcorner\kappa\urcorner)\vdash}\ \mathsf{LContr_W}$$

[11] The line labelled 'CT' abbreviates a Cut applied to $\vdash\kappa\otimes\mathsf{C}(\kappa)$ and $\kappa\otimes\mathsf{C}(\kappa)\vdash\mathsf{Ct}(\ulcorner\kappa\urcorner)$, which follows from the right-to-left direction of the schema CT of p. 6, namely $\vdash\mathsf{Ct}(\ulcorner\varphi\urcorner)\leftrightarrow(\varphi\otimes\mathsf{C}(\varphi))$.

Three copies of $\mathcal{D}_2$ can now be turned into a proof that $\mathsf{Ct}(\ulcorner\kappa\urcorner)$ is noncontractable, courtesy of C-L_1^+:

$$
\dfrac{
  \dfrac{
    \dfrac{
      \dfrac{
        \dfrac{
          \dfrac{\mathcal{D}_2}{\mathsf{LC}_{\mathsf{Ct}(\ulcorner\kappa\urcorner)}, \mathsf{Ct}(\ulcorner\kappa\urcorner) \vdash}
        }{\mathsf{LC}_{\mathsf{Ct}(\ulcorner\kappa\urcorner)} \vdash \neg\mathsf{Ct}(\ulcorner\kappa\urcorner)}\;\neg\text{-R}
      }{\mathsf{LC}_{\mathsf{Ct}(\ulcorner\kappa\urcorner)} \vdash \kappa}\;\text{Def. of }\kappa
    }{\mathsf{LC}_{\mathsf{Ct}(\ulcorner\kappa\urcorner)}, \mathsf{LC}_{\mathsf{Ct}(\ulcorner\kappa\urcorner)} \vdash \mathsf{C}(\kappa)}\;\mathsf{NEC}_{\mathsf{C}}
    \qquad
    \dfrac{
      \dfrac{
        \dfrac{\mathcal{D}_2}{\mathsf{LC}_{\mathsf{Ct}(\ulcorner\kappa\urcorner)}, \mathsf{Ct}(\ulcorner\kappa\urcorner) \vdash}
      }{\mathsf{LC}_{\mathsf{Ct}(\ulcorner\kappa\urcorner)} \vdash \neg\mathsf{Ct}(\ulcorner\kappa\urcorner)}\;\neg\text{-R}
    }{\mathsf{LC}_{\mathsf{Ct}(\ulcorner\kappa\urcorner)} \vdash \kappa}\;\text{Def. of }\kappa
  }{\mathsf{LC}_{\mathsf{Ct}(\ulcorner\kappa\urcorner)}, \mathsf{LC}_{\mathsf{Ct}(\ulcorner\kappa\urcorner)}, \mathsf{LC}_{\mathsf{Ct}(\ulcorner\kappa\urcorner)} \vdash \mathsf{Ct}(\ulcorner\kappa\urcorner)}\;\mathsf{CT}
  \qquad
  \dfrac{\mathcal{D}_2}{\mathsf{LC}_{\mathsf{Ct}(\ulcorner\kappa\urcorner)}, \mathsf{Ct}(\ulcorner\kappa\urcorner) \vdash}
}{
  \dfrac{
    \dfrac{\mathsf{LC}_{\mathsf{Ct}(\ulcorner\kappa\urcorner)}, \mathsf{LC}_{\mathsf{Ct}(\ulcorner\kappa\urcorner)}, \mathsf{LC}_{\mathsf{Ct}(\ulcorner\kappa\urcorner)}, \mathsf{LC}_{\mathsf{Ct}(\ulcorner\kappa\urcorner)} \vdash}{\mathsf{C}(\mathsf{Ct}(\ulcorner\kappa\urcorner)) \vdash}\;\text{C-}\mathsf{L}_1^+
  }{\vdash \neg\mathsf{C}(\mathsf{Ct}(\ulcorner\kappa\urcorner))}\;\neg\text{-R}
}\;\text{Cut}
$$

□

Since $\mathsf{Ct}(\ulcorner\kappa\urcorner)$ is provably non-contractable, it cannot be contracted on, and the Non-contractability Knower is blocked. If not all instances of structural contraction hold, contractable truth can be defined after all. Or can it?

4 Revenge

Our argument is in two steps. We first establish that κ is non-contractable. We then show that this very claim yields triviality.

Lemma 7. *Let S be any theory strong enough to prove the existence of a sentence κ equivalent to $\neg\mathsf{Ct}(\ulcorner\kappa\urcorner)$, with underlying logic at least as strong as* WMLL. *Then, S defines contractable truth only if S proves $\vdash \neg\mathsf{C}(\kappa)$.*

Proof. We notice that, by construction of κ and the WMLL-valid rule of double negation elimination,[12] $\neg\kappa$ entails $\mathsf{Ct}(\ulcorner\kappa\urcorner)$, and hence κ, so that both $\kappa \vdash \kappa$ and $\neg\kappa \vdash \kappa$ hold. Since $\kappa \oplus \neg\kappa$ is a theorem of WMLL, one can now infer $\kappa \oplus \kappa$ from $\kappa \oplus \neg\kappa$, courtesy of WPC (see p. 3). More formally:[13]

$$
\dfrac{
  \dfrac{}{\vdash \kappa \oplus \neg\kappa}\;\text{LEM}
  \qquad
  \dfrac{}{\kappa \vdash \kappa}\;\text{SRef}
  \qquad
  \dfrac{
    \dfrac{
      \dfrac{
        \dfrac{}{\neg\kappa \vdash \neg\kappa}\;\text{SRef}
      }{\neg\kappa \vdash \neg\neg\mathsf{Ct}(\ulcorner\kappa\urcorner)}\;\text{Def. of }\kappa
      \qquad
      \dfrac{}{\neg\neg\mathsf{Ct}(\ulcorner\kappa\urcorner) \vdash \mathsf{Ct}(\ulcorner\kappa\urcorner)}\;\neg\neg\text{-E}
    }{\neg\kappa \vdash \mathsf{Ct}(\ulcorner\kappa\urcorner)}\;\text{Cut}
  }{\neg\kappa \vdash \kappa}\;\text{FACT}
}{\vdash \kappa \oplus \kappa}\;\text{WPC}
$$

The above derivation, call it $\mathcal{D}_3$, can be used to derive $(\kappa \oplus \kappa) \to \kappa \vdash \kappa$:

$$
\dfrac{
  \dfrac{\mathcal{D}_3}{\vdash \kappa \oplus \kappa}
  \qquad
  \dfrac{}{\kappa \vdash \kappa}\;\text{SRef}
}{(\kappa \oplus \kappa) \to \kappa \vdash \kappa}\;\to\text{-L}
$$

[12] See Zardini (2011, Theorem 3.8, p. 514).

[13] The lines labelled 'LEM' and '$\neg\neg$-E' respectively abbreviate WMLL-proofs of $\kappa \oplus \neg\kappa$ and $\neg\neg\mathsf{Ct}(\ulcorner\kappa\urcorner) \vdash \mathsf{Ct}(\ulcorner\kappa\urcorner)$.

Call this derivation $\mathcal{D}_4$. We now use it to prove that κ is non-contractable:

$$
\dfrac{
\dfrac{\begin{matrix}\mathcal{D}_4\\ (\kappa\oplus\kappa)\to\kappa\vdash\kappa\end{matrix}}{(\kappa\oplus\kappa)\to\kappa\vdash\neg\mathsf{Ct}(\ulcorner\kappa\urcorner)}\;{\scriptstyle\text{Def. of }\kappa}
\qquad
\dfrac{\dfrac{\dfrac{
\begin{matrix}\mathcal{D}_4\\ (\kappa\oplus\kappa)\to\kappa\vdash\kappa\end{matrix}
\quad
\dfrac{\begin{matrix}\mathcal{D}_4\\ (\kappa\oplus\kappa)\to\kappa\vdash\kappa\end{matrix}}{(\kappa\oplus\kappa)\to\kappa,(\kappa\oplus\kappa)\to\kappa\vdash\mathsf{C}(\kappa)}\;{\scriptstyle\mathsf{NEC_C}}
}{[(\kappa\oplus\kappa)\to\kappa]^3\vdash\kappa\otimes\mathsf{C}(\kappa)}\;{\scriptstyle\otimes\text{-R}}}{[(\kappa\oplus\kappa)\to\kappa]^3\vdash\mathsf{Ct}(\ulcorner\kappa\urcorner)}\;{\scriptstyle\text{Def. of }\mathsf{Ct}(\ulcorner\kappa\urcorner)}}{\neg\mathsf{Ct}(\ulcorner\kappa\urcorner),[(\kappa\oplus\kappa)\to\kappa]^3\vdash}\;{\scriptstyle\neg\text{-L}}
}{
\dfrac{\dfrac{[(\kappa\oplus\kappa)\to\kappa]^4\vdash}{\mathsf{C}(\kappa)\vdash}\;{\scriptstyle\mathsf{C\text{-}L}_2^+}}{\vdash\neg\mathsf{C}(\kappa)}\;{\scriptstyle\neg\text{-R}}
}\;{\scriptstyle\text{Cut}}
$$

□

The claim that κ is non-contractable lands one into paradox once again, however. To see this, one notices in S that $\neg\mathsf{C}(\kappa)$ entails $\neg\mathsf{Ct}(\ulcorner\kappa\urcorner)$, and hence κ, whence $\vdash\mathsf{C}(\kappa)$ follows, courtesy of $\mathsf{NEC_C}$. But κ is non-contractable, i.e. $\neg\mathsf{C}(\kappa)$.

Proposition 8. *Let S be any theory strong enough to prove the existence of a sentence κ equivalent to $\neg\mathsf{Ct}(\ulcorner\kappa\urcorner)$, with underlying logic at least as strong as* WMLL. *Let $\vdash$ be S's consequence relation and suppose S defines contractable truth. Then, S is trivial.*

Proof. Let κ be a sentence provably equivalent to $\neg\mathsf{Ct}(\ulcorner\kappa\urcorner)$. Since, we're assuming, S defines contractable truth, by Lemma 7 S proves $\neg\mathsf{Ct}(\ulcorner\kappa\urcorner)$. We may then reason thus:[14]

$$
\dfrac{
\dfrac{}{\vdash\neg\mathsf{C}(\kappa)}\;{\scriptstyle\text{Lemma 7}}
\qquad
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\vdash\neg\mathsf{C}(\kappa)}\;{\scriptstyle\text{Lemma 7}}}{\vdash\neg\mathsf{Ct}(\ulcorner\kappa\urcorner)}\;{\scriptstyle\text{Def. of }\mathsf{Ct}(\ulcorner\kappa\urcorner)\text{ and logic}}}{\vdash\kappa}\;{\scriptstyle\text{Def. of }\kappa}}{\vdash\mathsf{C}(\kappa)}\;{\scriptstyle\mathsf{NEC_C}}}{\neg\mathsf{C}(\kappa)\vdash}\;{\scriptstyle\neg\text{-L}}
}{\vdash}\;{\scriptstyle\text{Cut}}
$$

□

This is bad news for the non-contractive theorist. Lemma 7 establishes that S defines contractable truth only if it proves $\neg\mathsf{C}(\kappa)$. But it follows from Proposition 8 that any such S proves $\neg\mathsf{C}(\kappa)$ only if it is trivial.

5 Concluding remarks

Our argument requires five main ingredients:

(i) FACT, to get $\vdash\kappa\oplus\kappa$;

(ii) that $\vdash\kappa$ and $\vdash\mathsf{C}(\kappa)$ yield $\vdash\mathsf{Ct}(\ulcorner\kappa\urcorner)$;

[14] The line labelled 'Def. of $\mathsf{Ct}(\ulcorner\kappa\urcorner)$ and logic' abbreviates the following passages: from $\neg\mathsf{C}(\kappa)$ to $\neg\mathsf{C}(\kappa)\oplus\neg\kappa$ (by right weakening and $\oplus$-R), then from $\neg\mathsf{C}(\kappa)\oplus\neg\kappa$ to $\neg(\mathsf{C}(\kappa)\otimes\kappa)$ (by the DeMorgan laws, which hold in WMLL), and finally from the latter to $\neg\mathsf{Ct}(\ulcorner\kappa\urcorner)$ by definition of Ct, i.e. the schema CT of p. 6 (see also footnote 11).

(iii) that S be closed under C-L_2^+, to get $\vdash \neg C(\kappa)$;

(iv) that $\neg C(\kappa) \vdash \neg Ct(\ulcorner\kappa\urcorner)$, to turn the claim that κ is non-contractable into an assertion of κ itself;

(v) that $\neg C(\kappa)$ yields triviality.

Items (i), (ii), and (iv) come straight from CT, the definition of $Ct(\ulcorner\kappa\urcorner)$, and some basic features of the logic WMLL. Item (iii) is motivated by the assumption that a sentence is non-contractable in S if and only if contracting on it in S (it doesn't matter how many times) makes S trivial. As for the claim that $\neg C(\kappa)$ yields triviality, namely (v), it turns on NEC_C. But, as Fact 3 shows, NEC_C is derivable from C-R, which is in turn justified by Theorem 1.

To be sure, we've assumed throughout that any adequate semantic theory should be non-hierarchical, in the sense of being able to consistently express meta-theoretical notions such as non-contractability. However, we submit, expressing in the object-language notions that have been traditionally formalised in a meta-theory is an integral part of the effort of treating truth and other fundamental semantical and logical concepts in one single language (Reinhart (1986, pp. 227-9); Field (2008, p. 18)). We conclude that the theory of contractable truth for a contraction-free theory S cannot be formulated in S.

References

Asenjo, F. and Tamburino, J.: 1975, Logic of antinomies, *Notre Dame Journal of Formal Logic* **16**, 17–44.

Beall, J.: 2006, True, false and paranormal, *Analysis* **66**(2), 102–14.

Beall, J.: 2007a, Truth and paradox: a philosophical sketch, *in* D. Jacquette (ed.), *Philosophy of Logic*, Elsevier, Oxford, pp. 325–410.

Beall, J.: 2009, *Spandrels of Truth*, Oxford University Press, Oxford.

Beall, J.: 2011, Multiple-conclusion LP and default classicality, *Review of Symbolic Logic* **4**(1), 326–336.

Beall, J. (ed.): 2007b, *Revenge of the Liar*, Oxford University Press.

Field, H.: 2007, Solving the paradoxes, escaping revenge, *in* J. Beall (ed.), *Revenge of the Liar*, Oxford University Press, pp. 53–144.

Field, H.: 2008, *Saving Truth from Paradox*, Oxford University Press, Oxford.

Fitch, F.: 1942, A basic logic, *Journal of Symbolic Logic* **7**, 105–14.

Fitch, F.: 1948, An extension of basic logic, *Journal of Symbolic Logic* **13**, 95–106.

Goodship, L.: 1996, On dialetheism, *Australasian Journal of Philosophy* **74**, 153–61.

Horsten, L.: 2009, Levity, *Mind* **118**(471), 555–581.

Kaplan, D. and Montague, R.: 1960, A paradox regained, *Notre Dame Journal of Formal Logic* **1**, 79–90.

Kripke, S.: 1975, Outline of a theory of truth, *Journal of Philosophy* **72**, 690–716.

Mares, E. and Paoli, F.: 2014, Logical consequence and the paradoxes, *Journal of Philosophical Logic* **43**, 439–469.

Myhill, J.: 1960, Some remarks on the notion of proof, *Journal of Philosophy* **57**(14), 461–71.

Myhill, J.: 1975, Levels of implication, *in* A. R. Anderson, R. C. Barcan-Marcus and R. M. Martin (eds), *The Logical Enterprise*, Yale University Press, New Haven, pp. 179–185.

Priest, G.: 2006, *Doubt Truth to be a Liar*, Oxford University Press, Oxford.

Priest, G.: 2007, Revenge, Field, and ZF, *in* J. Beall (ed.), *Revenge of the Liar*, Oxford University Press, pp. 225–233.

Reinhart, W. N.: 1986, Some remarks on extending and interpreting theories,with a partial predicate for truth, *Journal of Philosophical Logic* **15**, 219–51.

Scharp, K.: 2013, *Replacing Truth*, Oxford University Press, Oxford.

Shapiro, L.: 2010, Expressibility and the Liar's revenge, *Australasian Journal of Philosophy* pp. 1–18.

Shapiro, L.: 2011, Deflating logical consequence, *The Philosophical Quarterly* **61**, 320–42.

Zardini, E.: 2011, Truth without contra(di)ction, *Review of Symbolic Logic* **4**, 498–535.

Zardini, E.: 2013, Näive logical properties and structural properties, *The Journal of Philosophy* **110**(11), 633–44.

Zardini, E.: 2014, Naïve truth and naïve logical properties, *Review of Symbolic Logic* **7**(2), 351–384.

Zardini, E.: 2015, Getting one for two, or the contractors' bad deal. Towards a unified solution to the semantic paradoxes, *in* H. G. T. Achourioti F. Fujimoto and J. Martinez-Fernandez (eds), *Unifying the Philosophy of Truth*, Springer, pp. 461–93.

Zardini, E.: 2018, Instability and contraction. Forthcoming in *Journal of Philosophical Logic*.